

# Fortieth Anniversary of Extremal Projector Method for Lie Symmetries

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**ABSTRACT.** A brief review of the extremal projector method for Lie symmetries (Lie algebras and superalgebras as well as their quantum analogs) is given. A history of its discovery and some simplest applications are presented.

## 1. Introduction

In 1964 P.-O. Löwdin [L] first obtained an explicit expression of the extremal projection operator for the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ . Later such explicit formulas of the extremal projectors were found for all finite-dimensional simple Lie algebras [AS2, AST1, AST2, AST3], classical Lie superalgebras [T1], infinite-dimensional affine Kac-Moody algebras and superalgebras [T4], and also for their quantum ( $q$ -deformed) analogs [T5, KT]. At present the extremal projector method is a powerful and universal method for solving many problems in representation theory. For example, the method allows to classify irreducible modules, to decompose them on submodules (e.g. to analyze the structure of Verma modules), to describe reduced (super)algebras (which are connected with the reduction of a (super)algebra to a subalgebra), to construct bases of representations (e.g. the Gelfand-Tsetlin's type), to develop the detailed theory of Clebsch-Gordan coefficients and other elements of Wigner-Racah calculus (including compact analytic formulas for these elements and their symmetry properties) and so on.

In this paper we give a brief review of the extremal projector method for Lie symmetries (Lie algebras and superalgebras as well as their quantum analogs), namely, we provide a history of its discovery and some simplest applications.

## 2. Projection operators for finite and compact groups

Let  $G$  be a finite and compact group, and  $T$  be its representation in a linear space  $V$ , i.e.  $g \mapsto T(g)$ , ( $g \in G$ ), where  $T(g)$  are linear operators acting in  $V$ , satisfying  $T(g_1g_2) = T(g_1)T(g_2)$ . The representation  $T$  in  $V$  is irreducible if

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$\text{Lin}\{T(G)v\} = V$  for any nonzero vector  $v \in V$ . An irreducible representation (IR) is denoted by an additional upper index  $\lambda$ ,  $T^\lambda(g)$ , and also  $V^\lambda$ , or in matrix form:  $(T^\lambda(g)) = (t_{ij}^\lambda(g))$  ( $i, j = 1, 2, \dots, n$ ), where  $n$  is the dimension of  $V^\lambda$ .

It is well-known that the elements

$$(2.1) \quad P_{ij}^\lambda = \sum_{g \in G} T(g) t_{ij}^\lambda(g)$$

are projection operators for the finite group  $G$ , i.e. they satisfy the following properties:

$$(2.2) \quad P_{ij}^\lambda P_{kl}^{\lambda'} = \delta_{\lambda\lambda'} \delta_{jk} P_{il}^\lambda,$$

$$(2.3) \quad (P_{ij}^\lambda)^* = P_{ji}^\lambda,$$

where  $*$  is Hermitian conjugation.

The projection operators for a compact group  $G$  are modified as follows:

$$(2.4) \quad P_{ij}^\lambda = \int_{g \in G} T(g) t_{ij}^\lambda(g) dg.$$

In the case  $G = SO(3)$  (or  $SU(2)$ ) we have

$$(2.5) \quad P_{mm'}^j = \int T(\alpha, \beta, \gamma) D_{mm'}^j(\alpha, \beta, \gamma) \sin \beta d\alpha d\beta d\gamma,$$

where  $\alpha, \beta, \gamma$  are the Euler angles and  $D_{mm'}^j(\alpha, \beta, \gamma)$  is the Wigner  $D$ -function. The projection operator  $P^j := P_{jj}^j$  is called the projector on the highest weight  $j$ .

Thus we see that the projection operators in the form (2.1) or (2.4) require explicit expressions for the operator function  $T(g)$ , the matrix elements of IRs  $t_{ij}^\lambda(g)$ , and also (in the case of a compact group) the  $g$ -invariant measure  $dg$ . In the case of an arbitrary compact group  $G$  these expressions lead to several problems.

### 3. Löwdin-Shapiro extremal projector for the angular momentum Lie algebra

The angular momentum Lie algebra  $\mathfrak{so}(3)$  ( $\simeq \mathfrak{su}(2)$ ) is generated by the three elements (generators)  $J_+$ ,  $J_-$  and  $J_0$  with the defining relations:

$$(3.1) \quad \begin{aligned} [J_0, J_\pm] &= \pm J_\pm, & [J_+, J_-] &= 2J_0, \\ J_\pm^* &= J_\mp, & J_0^* &= J_0. \end{aligned}$$

The Casimir element  $\mathbf{C}_2$  of the angular momentum Lie algebra (or square of the angular momentum  $\mathbf{J}^2$ ) is given by:

$$(3.2) \quad \mathbf{C}_2 \equiv \mathbf{J}^2 = \frac{1}{2} (J_+ J_- + J_- J_+) + J_0^2 = J_- J_+ + J_0(J_0 + 1),$$

$$(3.3) \quad [J_i, \mathbf{J}^2] = 0.$$

Let  $\{|jm\rangle\}$  be the canonical basis of  $\mathfrak{su}(2)$ -IR corresponding to the spin  $j$  (wave functions with definite  $j$  and its projection  $m$  ( $m = -j, -j+1, \dots, j$ ), for example,

spherical harmonics,  $|jm\rangle \equiv Y_m^j$ ). These basis functions satisfy the relations:

$$(3.4) \quad \begin{aligned} \mathbf{J}^2|jm\rangle &= j(j+1)|jm\rangle, & J_0|jm\rangle &= m|jm\rangle, \\ J_{\pm}|jm\rangle &= \sqrt{(j \mp m)(j \pm m + 1)} |jm \pm 1\rangle. \end{aligned}$$

The vectors  $|jm\rangle$  can be represented as follows:

$$(3.5) \quad |jm\rangle = F_{m;j}^j |jj\rangle,$$

where

$$(3.6) \quad F_{m;j}^j = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}} J_-^{j-m} \quad \left( F_{j;m}^j := (F_{m;j}^j)^* = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}} J_+^{j-m} \right),$$

and  $|jj\rangle$  is the highest weight vector, i.e.

$$(3.7) \quad J_+|jj\rangle = 0.$$

It is obvious that the projector  $P^j$  on the highest weight, (2.5), satisfies the relations:

$$(3.8) \quad J_+P^j = P^jJ_- = 0, \quad (P^j)^2 = P^j.$$

An associative polynomial algebra of the generators  $J_{\pm}$ ,  $J_0$  is called the universal enveloping algebra of the angular momentum Lie algebra and it is denoted by  $U(\mathfrak{so}(3))$  (or  $U(\mathfrak{su}(2))$ ). The following proposition holds.

“No-go theorem”: *No nontrivial solution of the equations*

$$(3.9) \quad J_+P = PJ_- = 0$$

*exists in  $U(\mathfrak{su}(2))$ , i.e. a unique solution of these equations for  $P \in U(\mathfrak{su}(2))$  is trivial  $P \equiv 0$ .*<sup>1</sup>

Thus the theorem states that the projector  $P^j$  does not exist in the form of a polynomial of the generators  $J_{\pm}$ ,  $J_0$ . This no-go theorem was well known to mathematicians, but we can assume that it was not known to most physicists.

In 1964, exactly 40 years ago, the Swedish physicist and chemist P.-O. Löwdin<sup>2</sup>, who probably did not know the no-go theorem, published a paper in the journal *Rev. Mod. Phys.* [L], in which he considered the following operator:

$$(3.10) \quad P^j := \prod_{j' \neq j} \frac{\mathbf{J}^2 - j'(j' + 1)}{j(j + 1) - j'(j' + 1)}.$$

This element has the following properties. Let  $\Psi_{m=j}$  be an arbitrary eigenvector of the operator  $J_0$ :

$$(3.11) \quad J_0\Psi_m = m\Psi_m.$$

Due to completeness of the basis formed by the vectors  $\{|jm\rangle\}$  (for all possible spins  $j$  and their projections  $m$ ) the following expansion holds:<sup>3</sup>

$$(3.12) \quad \Psi_m = \sum_{j'} C_{j'} |j'm\rangle,$$

<sup>1</sup> A general mathematical statement of this theorem reads as follows: “*The universal enveloping algebra  $U(\mathfrak{g})$  of a simple Lie algebra  $\mathfrak{g}$  has no zero divisors.*”

<sup>2</sup> Per-Olov Löwdin was born in 1916 in Uppsala, Sweden, and died in 2000 (see <http://www.quantum-chemistry-history.com/Lowdin1.htm>).

<sup>3</sup> Here we consider a multiplicity free case.

and it is obvious that

$$(3.13) \quad P^j \Psi_{m=j} = C_j |jj\rangle .$$

From here we obtain the following properties of the element (3.10):

$$(3.14) \quad J_+ P^j = P^j J_- = 0 ,$$

$$(3.15) \quad [J_0, P^j] = 0 , \quad (P^j)^2 = P^j ,$$

provided that the left and right sides of these equalities act on vectors with a definite projection of angular momentum  $m = j$ . Therefore the element (3.10) is the projector on the highest weight.

After rather complicated calculations Löwdin reduced the operator (3.10) to the following form:

$$(3.16) \quad P^j = \sum_{n \geq 0} \frac{(-1)^n (2j+1)!}{n! (2j+n+1)!} J_-^n J_+^n .$$

One year later, in 1965, another physicist, J. Shapiro from USA, published a paper in *J. Math. Phys.* [Sh], in which he stated: “Let us forget the initial expression (3.10) and consider the defining relations (3.14) and (3.15), where  $P^j$  has the following ansatz:

$$(3.17) \quad P^j = \sum_{n \geq 0} C_n(j) J_-^n J_+^n . ”$$

Substituting this expression in (3.14) we directly obtain the formula (3.16).

We can remove the upper index  $j$  in  $P^j$  if we replace  $j \rightarrow J_0$ :

$$(3.18) \quad P = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_n(J_0) J_-^n J_+^n ,$$

$$(3.19) \quad \varphi_n(J_0) = \prod_{k=1}^n (2J_0 + k + 1)^{-1} .$$

The element  $P$  is called the extremal projector. If  $\Psi$  is an arbitrary function

$$(3.20) \quad \Psi = \sum_{j,m} C_{j,m} |jm\rangle ,$$

then

$$(3.21) \quad P\Psi = \sum_j C_{j,j} |jj\rangle .$$

The extremal projector  $P$  does not belong to  $U(\mathfrak{su}(2))$  but it belongs to some extension of the universal enveloping algebra. Let us determine this extension.

Consider the formal Taylor series

$$(3.22) \quad \sum_{n,k \geq 0} C_{n,k}(J_0) J_-^n J_+^k ,$$

where  $C_{n,k}(J_0)$  are rational functions of the Cartan element  $J_0$  and provided that for each series there exists a natural number  $N$  for which

$$(3.23) \quad |n - k| \leq N .$$

Let  $TU(\mathfrak{su}(2))$  be the linear space of such formal series. We can show that  $TU(\mathfrak{su}(2))$  is an associative algebra with respect to the multiplication of formal series. The associative algebra  $TU(\mathfrak{su}(2))$  is called the *Taylor extension* of  $U(\mathfrak{su}(2))$ . It is obvious that  $TU(\mathfrak{su}(2))$  contains  $U(\mathfrak{su}(2))$ .

*Remark.* The restriction (3.23) is important. Consider two series:

$$(3.24) \quad x_1 := \sum_{k \geq 0} J_+^k, \quad x_2 := \sum_{n \geq 0} J_-^n.$$

Their product is reduced to the form (3.22)

$$(3.25) \quad x_1 x_2 = \sum_{n, k \geq 0} \Delta_{n, k}(J_0) J_-^n J_+^k,$$

where  $\Delta_{n, k}(J_0)$  is not any rational function of  $J_0$ , and moreover it is a generalized function of  $J_0$ .

The extremal projector (3.18) belongs to the Taylor extension  $TU(\mathfrak{su}(2))$ . Therefore Löwdin and Shapiro found a solution of the equations (3.14), not in the space  $U(\mathfrak{su}(2))$ , but in its extension  $TU(\mathfrak{su}(2))$ .

Later Shapiro tried to generalize the obtained formula (3.18) to the case of  $\mathfrak{su}(3)$  ( $\mathfrak{u}(3)$ ). The Lie algebra  $\mathfrak{u}(3)$  is generated by 9 elements  $e_{ik}$  ( $i, k = 1, 2, 3$ ) with the relations:

$$(3.26) \quad [e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}, \quad e_{ij}^* = e_{ij}.$$

Shapiro considered the following ansatz for  $P$ :

$$(3.27) \quad P := \sum_{n_i, m_i \geq 0} C_{n_i, m_i}(e_{11}, e_{22}, e_{33}) e_{21}^{n_1} e_{31}^{n_2} e_{32}^{n_3} e_{12}^{m_1} e_{13}^{m_2} e_{23}^{m_3}$$

and he used the equations

$$(3.28) \quad e_{ij} P = P e_{ji} = 0 \quad (i < j),$$

$$(3.29) \quad [e_{ii}, P] = 0 \quad (i = 1, 2, 3).$$

From the last equation it follows that

$$(3.30) \quad \begin{cases} n_1 + n_2 = m_1 + m_2, \\ n_2 + n_3 = m_2 + m_3. \end{cases}$$

Under the conditions (3.30) the expression (3.27) belongs to  $TU(\mathfrak{su}(3))$ . A system of equations for the coefficients  $C_{n_i, m_i}(e_{11}, e_{22}, e_{33})$  was found to be too complicated, and Shapiro failed to solve this system.

In 1968 R.M. Asherova and Yu.F. Smirnov [AS1] made the first important step in order to obtain an explicit formula for the extremal projector for  $\mathfrak{u}(3)$ . They proposed to act with  $P$  described by the Shapiro ansatz (3.27) on the extremal projector of the subalgebra  $\mathfrak{su}(2)$  generated by the elements  $e_{23}$ ,  $e_{32}$ ,  $e_{22} - e_{33}$ ,

$$(3.31) \quad P_{23} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_n(e_{22} - e_{33}) e_{32}^n e_{23}^n,$$

where

$$(3.32) \quad \varphi_n(e_{22} - e_{33}) = \prod_{k=1}^n (e_{22} - e_{33} + k + 1)^{-1}.$$

Since  $e_{23}P_{23} = 0$ , therefore we obtain the following form for  $P(\mathfrak{su}(3))$ :

$$(3.33) \quad P = \sum_{n_i \geq 0} C_{n_1, n_2, n_3}(e_{11}, e_{22}, e_{33}) e_{21}^{n_1} e_{31}^{n_2 - n_3} e_{32}^{n_3} e_{12}^{n_1 - n_3} e_{13}^{n_2} P_{23} .$$

In this case the system of equations for the coefficients  $C_{n_i}(e_{ii})$  is simpler, and it was solved. However, the explicit expressions for the coefficients  $C_{n_i}(e_{ii})$  are rather complicated. The next simple idea [T] was to act on the expression (3.33) (from the left side) by the extremal projector of the subalgebra  $\mathfrak{su}(2)$  generated by the elements  $e_{12}, e_{21}, e_{11} - e_{22}$ . As a result we obtain the following simple form for  $P := P(\mathfrak{su}(3))$ :

$$(3.34) \quad P = P_{12} \left( \sum_{n \geq 0} C_n(e_{11} - e_{33}) e_{31}^n e_{13}^n \right) P_{23} .$$

The final formula is

$$(3.35) \quad P = P_{12} P_{13} P_{23} ,$$

where

$$(3.36) \quad P_{ij} = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_n(e_{ii} - e_{jj}) e_{ji}^n e_{ij}^n \quad (i < j) ,$$

$$(3.37) \quad \varphi_n(e_{ii} - e_{jj}) = \prod_{k=1}^n (e_{ii} - e_{jj} + k + j - i)^{-1} .$$

It was found that this formula is fundamental. In the next section we give the explicit formula of the extremal projector for all finite-dimensional simple Lie algebras.

#### 4. Extremal projector for simple Lie algebras

Let  $\mathfrak{g}$  be a finite-dimensional simple Lie algebra and  $\Delta_+(\mathfrak{g})$  be its positive root system. A generalization of the formulas (3.35)–(3.37) to the case of  $\mathfrak{g}$  is connected with the notion of normal ordering in the system  $\Delta_+(\mathfrak{g})$ .

**DEFINITION 4.1.** We say that the system  $\Delta_+(\mathfrak{g})$  is in normal ordering if each composite (not simple) root  $\gamma = \alpha + \beta$  ( $\alpha, \beta, \gamma \in \Delta_+(\mathfrak{g})$ ) is written between its constituents  $\alpha$  and  $\beta$ .

*Remarks.* (i) We can show that the normal ordering in the system  $\Delta_+(\mathfrak{g})$  exists (see [T6]). (ii) For classical simple Lie superalgebras the normal ordering is defined for a reduced root system  $\underline{\Delta}_+(\mathfrak{g})$  [T1], and for infinite-dimensional affine Kac–Moody algebras and superalgebras the definition of the normal ordering is modified [T4, T5, T7].

The normally ordered system  $\Delta_+(\mathfrak{g})$  is denoted by the symbol  $\vec{\Delta}_+(\mathfrak{g})$ . Let  $e_{\pm\gamma}, h_\gamma$  be Cartan–Weyl root vectors normalized by the condition

$$(4.1) \quad [e_\gamma, e_{-\gamma}] = h_\gamma .$$

**THEOREM 4.2.** *The equations*

$$(4.2) \quad e_\gamma P = P e_{-\gamma} = 0 \quad (\forall \gamma \in \Delta_+(\mathfrak{g})) , \quad P^2 = P$$

have a unique nonzero solution in the space of the Taylor extension  $T_q(\mathfrak{g})$  and this solution has the form

$$(4.3) \quad P = \prod_{\gamma \in \tilde{\Delta}_+(\mathfrak{g})} P_\gamma ,$$

where the elements  $P_\gamma$  are defined by the formulae

$$(4.4) \quad P_\gamma = \sum_{n \geq 0} \frac{(-1)^n}{n!} \varphi_{\gamma,n} e_{-\gamma}^m e_\gamma^m ,$$

$$(4.5) \quad \varphi_{\gamma,n} = \prod_{k=1}^n \left( h_\gamma + (\rho, \gamma) + \frac{1}{2}(\gamma, \gamma)k \right)^{-1} .$$

Here  $\rho$  is the half-sum of all positive roots.

In the next Sections 5–8 we consider some simplest applications of the extremal projectors for the Lie algebras  $\mathfrak{su}(2)$  and  $\mathfrak{su}(3)$ .

## 5. Clebsch-Gordan coefficients for the angular momentum Lie algebra

Let  $\{|j_i m_i\rangle\}$  be wave functions of two systems (canonical bases of two IRs of the angular momentum Lie algebra) with spins  $j_i$  ( $i = 1, 2$ ). Then  $\{|j_1 m_1\rangle |j_2 m_2\rangle\}$  is the uncoupled (tensor) basis in the representation  $j_1 \otimes j_2$  for  $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ . In this representation there is another basis  $|j_1 j_2 : j_3 m_3\rangle$  which is called coupled with respect to  $\mathfrak{su}(2)$  (which is a diagonal embedding in  $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ ). We can extend the coupled basis in terms of the uncoupled one:

$$(5.1) \quad |j_1 j_2 : j_3 m_3\rangle = \sum_{m_1, m_2} (j_1 m_1 j_2 m_2 | j_3 m_3) |j_1 m_1\rangle |j_2 m_2\rangle ,$$

where matrix elements  $(j_1 m_1 j_2 m_2 | j_3 m_3)$  are called Clebsch-Gordan coefficients (CGC).

One can show that CGC can be presented in the form:

$$(5.2) \quad (j_1 m_1 j_2 m_2 | j_3 m_3) = \frac{\langle j_1 m_1 | \langle j_2 m_2 | P_{m_3; j_3}^{j_3} | j_1 j_1 \rangle | j_2 j_3 - j_1 \rangle}{\sqrt{\langle j_1 j_1 | \langle j_2 j_3 - j_1 | P_{j_3; j_3}^{j_3} | j_1 j_1 \rangle | j_2 j_3 - j_1 \rangle}}$$

where  $P_{m_3; m'_3}^{j_3}$  is a general projection operator which is connected with the extremal projector as follows:

$$(5.3) \quad P_{m_3; m'_3}^{j_3} := F_{m_3; j_3}^{j_3} P_{j_3; m'_3}^{j_3} F_{j_3; m'_3}^{j_3} ,$$

and it is constructed from the generators of the coupled system,  $J_i(3) = J_i(1) + J_i(2) \equiv \Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i$ , ( $i = \pm, 0$ ).

Using the explicit formulas (3.6) and (3.16) one can easily obtain the final formula of  $\mathfrak{su}(2)$ -CGC:

$$(5.4) \quad \begin{aligned} (j_1 m_1 j_2 m_2 | j_3 m_3) &= \delta_{m_1+m_2, m_3} \\ &\times \sqrt{\frac{(2j_3+1)(j_2-m_2)!(j_3+m_3)!(j_1+j_2+j_3+1)!(j_1+j_2-j_3)!(j_1-j_2+j_3)!}{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_3-m_3)!(-j_1+j_2+j_3)!}} \\ &\times \sum_n \frac{(-1)^{j_1+j_2-j_3-n} (2j_2-n)!(j_1+j_2-m_3-n)!}{n!(j_2-m_2-n)!(j_1+j_2-j_3-n)!(j_1+j_2+j_3+1-n)!} . \end{aligned}$$

This formula was obtained by Shapiro [Sh], and it allows to obtain all classical symmetry properties of CGC: permutations and conjugations.

## 6. Gelfand-Tsetlin basis for $\mathfrak{su}(3)$

The Lie algebra  $\mathfrak{su}(3)$  is generated by the elements  $e_{ij}$  ( $i, j = 1, 2, 3$ ) provided  $e_{11} + e_{22} + e_{33} = 0$  with the relations (3.26). Let  $(\lambda\mu)$  be a finite-dimensional irreducible representation (IR) of  $\mathfrak{su}(3)$  with the highest weight  $(\lambda\mu)$  ( $\lambda$  and  $\mu$  are nonnegative integers). The highest weight vector, denoted by the symbol  $|(\lambda\mu)h\rangle$ , satisfies the relations

$$\begin{aligned} (6.1) \quad & (e_{11} - e_{22})|(\lambda\mu)h\rangle = \lambda|(\lambda\mu)h\rangle, \\ & (e_{22} - e_{33})|(\lambda\mu)h\rangle = \mu|(\lambda\mu)h\rangle, \\ & e_{ij}|(\lambda\mu)h\rangle = 0 \quad (i < j). \end{aligned}$$

The labelling of other basis vectors in IR  $(\lambda\mu)$  depends upon the choice of subalgebras of  $\mathfrak{su}(3)$  (or in another words, depends upon which reduction chain from  $\mathfrak{su}(3)$  to the subalgebras is chosen). Here we use the Gelfand-Tsetlin reduction chain:

$$(6.2) \quad \mathfrak{su}(3) \supset u_Y(1) \otimes \mathfrak{su}_T(2) \supset u_{T_0}(1),$$

where the subalgebra  $\mathfrak{su}_T(2)$  is generated by the elements

$$(6.3) \quad T_+ := e_{23}, \quad T_- := e_{32}, \quad T_0 := \frac{1}{2}(e_{22} - e_{33}),$$

the subalgebra  $u_{T_0}(1)$  is generated by  $T_0$ , and  $u_Y(1)$  is generated by  $Y$ <sup>4</sup>

$$(6.4) \quad Y = -\frac{1}{3}(2e_{11} - e_2 - e_3).$$

In the case of the reduction chain (6.2) the basis vectors of IR  $(\lambda\mu)$  are denoted by

$$(6.5) \quad |(\lambda\mu)jtt_z\rangle.$$

Here the set  $jtt_z$  characterizes the hypercharge  $Y$  and the T-spin and its projection:

$$\begin{aligned} (6.6) \quad & Y|(\lambda\mu)jtt_z\rangle = y|(\lambda\mu)jtt_z\rangle, \quad T_0|(\lambda\mu)jtt_z\rangle = t_z|(\lambda\mu)jtt_z\rangle, \\ & T_{\pm}|(\lambda\mu)jtt_z\rangle = \sqrt{(t \mp t_z)(t \pm t_z + 1)}|(\lambda\mu)jtt_z \pm 1\rangle, \end{aligned}$$

where the parameter  $j$  is connected with the eigenvalue  $y$  of the operator  $Y$  as follows:  $y = -\frac{1}{3}(2\lambda + \mu) + 2j$ . It is not hard to show (see [PST1, AST5]) that the orthonormalized vectors (6.6) can be represented in the following form:

$$(6.7) \quad |(\lambda\mu)jtt_z\rangle = F_-^{(\lambda\mu)}(jtt_z)|(\lambda\mu)h\rangle := N_{jt}^{(\lambda\mu)} P_{t_z; t}^t e_{31}^{j+\frac{1}{2}\mu-t} e_{21}^{j-\frac{1}{2}\mu+t} |(\lambda\mu)h\rangle,$$

where  $P_{t_z; t}^t$  is the general projection operator of the type (5.3) for the Lie algebra  $\mathfrak{su}_T(2)$ , and the normalization factor  $N_{jt}^{(\lambda\mu)}$  has the form

$$(6.8) \quad N_{jt}^{(\lambda\mu)} = \left( \frac{(\lambda + \frac{1}{2}\mu - j + t + 1)!(\lambda + \frac{1}{2}\mu - j - t)!(\frac{1}{2}\mu + j + t + 1)!(\frac{1}{2}\mu - j + t)!}{\lambda! \mu! (\lambda + \mu + 1)!(j + \frac{1}{2}\mu - t)!(j - \frac{1}{2}\mu + t)!(2t + 1)!} \right)^{\frac{1}{2}}.$$

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<sup>4</sup> In elementary particle theory the subalgebra  $\mathfrak{su}_T(2)$  is called the T-spin algebra and the element  $Y$  is called the hypercharge operator.



The quantum numbers  $jt$  take all nonnegative integers and half-integers such that the sum  $\frac{1}{2}\mu + j + t$  is an integer and they are subjected to the constraint

$$(6.9) \quad \begin{cases} \frac{1}{2}\mu + j - t \geq 0, & -\frac{1}{2}\mu + j + t \geq 0, \\ \frac{1}{2}\mu - j + t \geq 0, & \frac{1}{2}\mu + j + t \leq \lambda + \mu. \end{cases}$$

For every fixed  $t$  the projection  $t_z$  takes the values  $t_z = -t, -t+1, \dots, t-1, t$ .

The explicit form (6.7) of the basis vectors  $\{ |(\lambda\mu)jtt_z\rangle \}$  allows to calculate easily the actions of the generators  $e_{ij}$  (see [PST1, AST5]).

### 7. Tensor form of the $\mathfrak{su}(3)$ projection operator

It is obvious that the extremal projector of  $\mathfrak{su}(3)$  can be presented in the form

$$(7.1) \quad P(\mathfrak{su}(3)) = P(\mathfrak{su}_T(2))(P_{12}P_{13})P(\mathfrak{su}_T(2)).$$

Now we present the middle part of (7.1) in terms of the  $\mathfrak{su}_T(2)$  tensor operators. To this end, we substitute the explicit expression for the factors  $P_{12}$  and  $P_{13}$ , and combine monomials  $e_{21}^n e_{31}^m$  and  $e_{12}^n e_{13}^m$ . After some manipulations with sums we obtain the following expression for the extremal projection operator  $P := P(\mathfrak{su}(3))$  in terms of tensor operators:

$$(7.2) \quad P = P(\mathfrak{su}_T(2)) \left( \sum_{jj_z} A_{jj_z} \tilde{R}_{j_z}^j R_{j_z}^j \right) P(\mathfrak{su}_T(2))$$

where

$$(7.3) \quad A_{jj_z} = \frac{(-1)^{3j} \varphi_{12} (\varphi_{12} + j + j_z - 1)! (\varphi_{13})!}{(2j)! (\varphi_{12} + 2j)! (\varphi_{13} + j + j_z)!},$$

$\varphi_{1i+1} := e_{11} - e_{i+1i+1} + i$ , ( $i = 1, 2$ ), and

$$(7.4) \quad \tilde{R}_{j_z}^j = \sqrt{\frac{(2j)!}{(j-j_z)!(j+j_z)!}} e_{21}^{j+j_z} e_{31}^{j-j_z},$$

$$(7.5) \quad R_{j_z}^j = \sqrt{\frac{(2j)!}{(j-j_z)!(j+j_z)!}} e_{12}^{j-j_z} e_{13}^{j+j_z}.$$

The elements  $R_{j_z}^j$  and  $\tilde{R}_{j_z}^j$  are irreducible tensor components, i.e. they satisfy the relations

$$(7.6) \quad [T_i, R_{j_z}^j] = \sum_{j'_z} \langle jj'_z | T_i | jj_z \rangle R_{j'_z}^j.$$

Below we assume that the  $\mathfrak{su}(3)$  extremal projection operator (7.2) acts in a weight space with the weight  $(\lambda\mu)$  and in this case the symbol  $P$  is supplied with the index  $(\lambda\mu)$ ,  $P^{(\lambda\mu)}$ , and all the Cartan elements  $e_{ii} - e_{i+1i+1}$  on the right side of (7.2) are replaced by the corresponding weight components  $\lambda$  and  $\mu$ .

Now we multiply the projector  $P^{(\lambda\mu)}$  from the left side by the lowering operator  $F^{(\lambda\mu)}(jtt_z)$  and from the right side by the rising operator  $(F^{(\lambda\mu)}(jtt_z))^*$ , and we finally find the tensor form of the general  $\mathfrak{su}(3)$  projection operator:

$$(7.7) \quad P_{jtt_z; j't't'_z}^{(\lambda\mu)} = \sum_{j''t''} B_{j''t''}^{(\lambda\mu)} \widetilde{\mathbf{R}}_{tt_z, t''t''}^{j+j''} \mathbf{R}_{t''t'', t't'_z}^{j''+j'}.$$

were the coefficients  $B_{j''t''}^{(\lambda\mu)}$  are given by

$$\begin{aligned}
 B_{j''t''}^{(\lambda\mu)} &= \frac{(-1)^{2j+j'+j''-t'+t''}(\lambda+1)(\mu+1)(\lambda+\mu+2)}{(\lambda+\frac{1}{2}\mu+j''+t''+2)!(\lambda+\frac{1}{2}\mu+j''-t''+1)!(2j'')!} \begin{Bmatrix} j & j'' & j+j'' \\ t'' & t & \frac{1}{2}\mu \end{Bmatrix} \\
 (7.8) \quad &\times \begin{Bmatrix} j' & j'' & j'+j'' \\ t'' & t' & \frac{1}{2}\mu \end{Bmatrix} \sqrt{(\lambda+\frac{1}{2}\mu-j+t+1)!(\lambda+\frac{1}{2}\mu-j-t)!} \\
 &\times \left( \frac{(\lambda+\frac{1}{2}\mu-j'+t'+1)!(\lambda+\frac{1}{2}\mu-j'-t')!(2j+2j''+1)(2j'+2j''+1)}{(2j)!(2j')!(2t+1)(2t'+1)} \right)^{\frac{1}{2}}.
 \end{aligned}$$

The operators  $\widetilde{\mathbf{R}}_{tt_z, t''t_z'}^{j+j''}$  and  $\mathbf{R}_{t''t_z', t't_z}^{j''+j'}$  are given by

$$(7.9) \quad \mathbf{R}_{tt_z, t't_z'}^j := \sqrt{(2t+1)} \sum_{j_z t_z''} (j j_z t t_z'' | t' t_z') R_{j_z}^j P_{t_z'', t_z}^{t_z}.$$

The formula (7.7) is the key for the calculation of  $\mathfrak{su}(3)$ -Clebsch-Gordan coefficients.

### 8. General form of Clebsch-Gordan coefficients for $\mathfrak{su}(3)$

For convenience we introduce the short notations  $\Lambda := (\lambda\mu)$  and  $\gamma := j t t_z$ , and therefore the basis vector  $|(\lambda\mu) j t t_z\rangle$  will be denoted by  $|\Lambda\gamma\rangle$ . Let  $\{|\Lambda_i\gamma_i\rangle\}$  denote bases of two IRs  $\Lambda_i$  ( $i = 1, 2$ ). Then  $\{|\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle\}$  form a basis in the representation  $\Lambda_1 \otimes \Lambda_2$  of  $\mathfrak{su}(3) \otimes \mathfrak{su}(3)$ . In such a representation there is another coupled basis  $|\Lambda_1\Lambda_2: s\Lambda_3\gamma_3\rangle$ , where the index  $s$  classifies the multiplicity of the representations  $\Lambda_3$ . We can expand the coupled basis in terms of the tensor (“uncoupled”) basis  $\{|\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle\}$ :

$$(8.1) \quad |\Lambda_1\Lambda_2: s\Lambda_3\gamma_3\rangle = \sum_{\gamma_1, \gamma_2} (\Lambda_1\gamma_1 \Lambda_2\gamma_2 | s\Lambda_3\gamma_3) |\Lambda_1\gamma_1\rangle |\Lambda_2\gamma_2\rangle,$$

where the matrix element  $(\Lambda_1\gamma_1 \Lambda_2\gamma_2 | s\Lambda_3\gamma_3)$  is the Clebsch-Gordan coefficient of  $\mathfrak{su}(3)$ .

We can show that any CGC of  $\mathfrak{su}(3)$  can be represented in terms of a linear combination of the matrix elements of the projection operator (7.7)

$$(8.2) \quad (\Lambda_1\gamma_1 \Lambda_2\gamma_2 | s\Lambda_3\gamma_3) = \sum_{\gamma'_2} C(\gamma'_2) \langle \Lambda_1\gamma_1 | \langle \Lambda_2\gamma_2 | P_{\gamma_3, h}^{\Lambda_3} | \Lambda_1 h \rangle | \Lambda_2\gamma'_2 \rangle.$$

Classification of the multiple representations  $\Lambda_3$  in the representation  $\Lambda_1 \otimes \Lambda_2$  is a special problem and we shall not touch it here.

We give here an explicit expression for the more general matrix elements in comparison with the right-side of (8.2):

$$(8.3) \quad \langle \Lambda_1\gamma_1 | \langle \Lambda_2\gamma_2 | P_{\gamma_3, \gamma'_3}^{\Lambda_3} | \Lambda_1\gamma'_1 \rangle | \Lambda_2\gamma'_2 \rangle.$$

Using (7.7)–(7.9) and the Wigner-Racah calculus for the subalgebra  $\mathfrak{su}(2)$  it is not hard to obtain the following result (see [PST1, AST5]):

$$\begin{aligned}
 \langle \Lambda_1 \gamma_1 | \langle \Lambda_2 \gamma_2 | P_{\gamma_3, \gamma'_3}^{\Lambda_3} | \Lambda_1 \gamma'_1 \rangle | \Lambda_2 \gamma'_2 \rangle &= (t_1 t_{1z} t_2 t_{2z} | t_3 t_{3z}) (t_1 t'_{1z} t_2 t'_{2z} | t'_3 t'_{3z}) \\
 &\times (\lambda_3 + 1)(\mu_3 + 1)(\lambda_3 + \mu_3 + 2) A \sum_{j'_1 j'_2 t'_1 t'_2 t'_3} C_{j'_1 j'_2 t'_1 t'_2 t'_3} \\
 &\times \begin{Bmatrix} j_1 - j''_1 & j_2 - j''_2 & j_1 + j_2 - j''_1 - j''_2 \\ t''_1 & t''_2 & t''_3 \\ t_1 & t_2 & t_3 \end{Bmatrix} \begin{Bmatrix} j'_1 - j''_1 & j'_2 - j''_2 & j'_1 + j'_2 - j''_1 - j''_2 \\ t''_1 & t''_2 & t''_3 \\ t'_1 & t'_2 & t'_3 \end{Bmatrix},
 \end{aligned} \tag{8.4}$$

where

$$\begin{aligned}
 A &= \left( \frac{(2t_1+1)(2t_2+1)(2j_1+1)!(2j_2+1)!(\lambda_3+\frac{1}{2}\mu_3-j_3+t_3+1)!(\lambda_3+\frac{1}{2}\mu_3-j_3-t_3)!}{(\lambda_1+\frac{1}{2}\mu_1-j_1+t_1+1)!(\lambda_1+\frac{1}{2}\mu_1-j_1-t_1)!(\lambda_2+\frac{1}{2}\mu_2-j_2+t_2+1)!(\lambda_2+\frac{1}{2}\mu_2-j_2-t_2)!(2j_3)!} \right. \\
 &\times \left. \frac{(2t'_1+1)(2t'_2+1)(2j'_1+1)!(2j'_2+1)!(\lambda_3+\frac{1}{2}\mu_3-j'_3+t'_3+1)!(\lambda_3+\frac{1}{2}\mu_3-j'_3-t'_3)!}{(\lambda_1+\frac{1}{2}\mu_1-j'_1+t'_1+1)!(\lambda_1+\frac{1}{2}\mu_1-j'_1-t'_1)!(\lambda_2+\frac{1}{2}\mu_2-j'_2+t'_2+1)!(\lambda_2+\frac{1}{2}\mu_2-j'_2-t'_2)!(2j'_3)!} \right)^{\frac{1}{2}},
 \end{aligned} \tag{8.5}$$

$$\begin{aligned}
 C_{j'_1 j'_2 t'_1 t'_2 t'_3} &= \frac{(-1)^{2(j_1+j_2+j'_3-j''_1-j''_2)} (2(j_1+j_2-j''_1-j''_2)+1)!}{(2j''_1)!(2j''_2)!(2j_1-2j''_1)!(2j_2-2j''_2)!(2j'_1-2j''_2)!} \\
 &\times \frac{(2(j'_1+j'_2-j''_1-j''_2)+1)!(2t''_1+1)(2t''_2+1)(2t''_3+1)}{(2j'_2-2j''_2)!(2(j_1+j_2-j_3-j''_1-j''_2))!} \\
 &\times \frac{(\lambda_1+\frac{1}{2}\mu_1-j''_1+t''_1+1)!(\lambda_1+\frac{1}{2}\mu_1-j''_1-t''_1)!(\lambda_2+\frac{1}{2}\mu_2-j''_2+t''_2+1)!(\lambda_2+\frac{1}{2}\mu_2-j''_2-t''_2)!}{(\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j''_1-j''_2+t''_3+2)!(\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j''_1-j''_2-t''_3+1)!} \\
 &\times \begin{Bmatrix} j_1 - j''_1 & j''_1 & j_1 \\ \frac{1}{2}\mu_1 & t_1 & t''_1 \end{Bmatrix} \begin{Bmatrix} j_2 - j''_2 & j''_2 & j_2 \\ \frac{1}{2}\mu_2 & t_2 & t''_2 \end{Bmatrix} \begin{Bmatrix} j_3 & j_1 + j_2 - j_3 - j''_1 - j''_2 & j_1 + j_2 - j''_1 - j''_2 \\ t_3 & t_3 & \frac{1}{2}\mu_3 \end{Bmatrix} \\
 &\times \begin{Bmatrix} j'_1 - j''_1 & j''_1 & j'_1 \\ \frac{1}{2}\mu_1 & t'_1 & t''_1 \end{Bmatrix} \begin{Bmatrix} j'_2 - j''_2 & j''_2 & j'_2 \\ \frac{1}{2}\mu_2 & t'_2 & t''_2 \end{Bmatrix} \begin{Bmatrix} j'_3 & j'_1 + j'_2 - j'_3 - j''_1 - j''_2 & j'_1 + j'_2 - j''_1 - j''_2 \\ t'_3 & t'_3 & \frac{1}{2}\mu_3 \end{Bmatrix}
 \end{aligned} \tag{8.6}$$

Here everywhere the braces denote  $6j$ - and  $9j$ -symbols of the Lie algebra  $\mathfrak{su}(2)$ .

## 9. Bibliographical notes of applications for the extremal projectors

For the convenience of the reader the main development of the subject will be characterized in this Section by the most important references. In particular:

- Explicit description of irreducible representations of (super)algebras (construction of different bases, actions of generators and their properties).

*Results:* The Gel'fand-Tsetlin bases for:

$\mathfrak{su}(n)$  (R.M. Asherova, Yu.F. Smirnov and V.N. Tolstoy (1973)),

$\mathfrak{so}(n)$  (V.N. Tolstoy (1975, unpublished)),

$G_2$  (D.T. Sviridov, Yu.F. Smirnov and V.N. Tolstoy (1976)),

$\mathfrak{osp}(1|2)$  (F.A. Berezin and V.N. Tolstoy (1980)),

$\mathfrak{gl}(m|n)$  (V.N. Tolstoy, I.F. Istomina and Yu.F. Smirnov (1986)),

$U_q(\mathfrak{su}(n))$  (V.N. Tolstoy (1990)),

$U_q(\mathfrak{su}(1|n))$  (T.D. Palev and V.N. Tolstoy (1991)),

$\mathfrak{sp}(2n)$  (A.I. Molev (1999)).

- The theory of Clebsch-Gordan coefficients of the simple Lie algebras.  
*Results:*  
 $\mathfrak{su}(3)$  (Z. Pluhar, Yu.F. Smirnov and V.N. Tolstoy (1981-86)),  
 $U_q(\mathfrak{su}(2))$  (Yu.F. Smirnov, V.N. Tolstoy and Yu.I. Khatritonov (1991-93)),  
 $U_q(\mathfrak{su}(3))$  (R.M. Asherova, Yu.F. Smirnov V.N. Tolstoy (2001)),  
 $U_q(\mathfrak{su}(n))$  (V.N. Tolstoy and D.J. Draayer (2000)).
- Description of reduction algebras (Mikelson's algebras).  
*Results:*  $A_n, B_n, C_n, D_n$  (D.P. Zhelobenko (1983)),  
 $\mathfrak{su}(m|n), \mathfrak{osp}(m|2n)$  (V.N. Tolstoy (1986)),  
 $U_q(\mathfrak{su}(n))$  (V.N. Tolstoy (1990)),  
 $U_q(\mathfrak{su}(1|n))$  (T.D. Palev and V.N. Tolstoy (1991)).
- Description of Verma modules of Lie (super)algebras (singular vectors and their properties).  
*Results:* for the simple Lie algebras (D.P. Zhelobenko (1985)).
- Construction of solutions of the Yang-Baxter equation with the help of projection operators.  
*Results:* for  $\mathfrak{u}(3)$  and  $\mathfrak{u}(n)$  (Yu.F. Smirnov and V.N. Tolstoy (1990); V. Tarasov and A. Varchenko (2002)).
- Connection between extremal projectors and integral projection operators.  
*Results:* for the simple Lie algebras (A.N. Leznov and M.V. Savel'ev (1974)).
- Connection between extremal projectors and canonical elements.  
*Results:* for  $q$ -boson Kashiwara algebras (T. Nakashima (2004)).
- Generalization of extremal projectors.  
*Results:* for  $\mathfrak{sl}(2)$  (V.N. Tolstoy (1988)),  
 $U_q(\mathfrak{sl}(2))$  (H.-D. Doebner and V.N. Tolstoy (1996)).
- Construction of indecomposable representations.  
*Results:* for  $U_q(\mathfrak{sl}(2))$  (H.-D. Doebner and V.N. Tolstoy (1996)).

## References

- [AS1] R.M. Asherova and Yu.F. Smirnov, *Projection operators and Clebsch-Gordan coefficients for the group  $SU(3)$* , Nucl. Phys. B 4 (1968), 399–412
- [AS2] ———, *Projection operators for classical groups* (Russian), Uspekhi Mat. Nauk 24 (1969), 227–228
- [AST1] R.M. Asherova, Yu.F. Smirnov, and V.N. Tolstoy, *Projection operators for the simple Lie groups*, Teor. Mat. Fiz. 8 (1971), 255–271 (in Russian)
- [AST2] ———, *Projection operators for the simple Lie groups. II. General scheme for construction of lowering operators. The case of the group  $SU(n)$* , Teor. Mat. Fiz. 15 (1973), 107–119 (in Russian)
- [AST3] ———, *A description of some class of projection operators for semisimple complex Lie algebras*, Matem. Zametki 26 (1979), 15–25 (in Russian)
- [AST4] ———, *Weyl  $q$ -coefficients for  $u_q(3)$  and Racah  $q$ -coefficients, for  $su_q(2)$* , Sov. Phys. Atom. Nucl. 59 10 (1996), 1795–1807 [Yadernaya Fiz. 59 (1996), 1859–1872]
- [AST5] ———, *On a general analytical formula for the  $U_q(\mathfrak{su}(3))$  Clebsch-Gordan coefficients*, Sov. Phys. Atom. Nucl. 64 (2001), 2080–2085 [math.QA/0103187]

- [BT] F.A. Berezin and V.N. Tolstoy, *The group with Grassmann structure  $UOSP(1|2)$* , Commun. Math. Phys. 78 (1981), 409–428
- [DT] H.-D. Doebner and V.N. Tolstoy, *Adjoint “extremal projectors” and indecomposable representations for  $U_q(sl(2, \mathbb{C}))$* , in: *Quantum symmetries* (H.-D. Doebner and V.K. Dobrev, eds.), World Scientific, River Edge NJ 1993, pp. 229–245
- [KT] S.M. Khoroshkin and V.N. Tolstoy, *Extremal projector and universal  $R$ -matrix for quantum contragredient Lie (super)algebras*, in: *Quantum Groups and Related Topics* (R. Gielerak et al., eds.), Kluwer Academic Publishers, Dordrecht 1992, pp. 23–32
- [KST1] V.A. Knyr, Yu.F. Smirnov, and V.N. Tolstoy, *On the “vector coupling” of unitary spins*, Rep. Math. Phys. 8 (1975) 343–356
- [KST2] ———, *Analysis of irreducible representations of the group  $U(4) \supset U(2) \otimes U(2)$  by the projection operator method*, Rep. Math. Phys. 20 (1984), 347–355
- [LS] A.N. Leznov and M.V. Savelev, *About one parametrization of compact groups*, Funct. Anal. Appl. 8 (1974), 347–348 [Funkts. Anal. Prilozh. 8 (1974) 87–88]
- [L] P.-O. Löwdin, *Angular momentum wavefunctions constructed by projector operators*, Rev. Mod. Phys. 36, (1964) 966–976
- [M] A.I. Molev, *A basis for representations of symplectic Lie algebras*, Commun. Math. Phys. 201 (1999), 591–618
- [N] T. Nakashima, *Extremal projectors of  $q$ -boson algebras*, Commun. Math. Phys. 244 (2004), 285–296
- [PT] T.D. Palev and V.N. Tolstoy, *Finite-dimensional irreducible representations of the quantum superalgebra  $U_q(gl(n/1))$* , Commun. Math. Phys. 141 (1991), 549–558
- [PST1] Z. Pluhar, Yu.F. Smirnov, and V.N. Tolstoy, *A novel approach to the  $SU(3)$  symmetry calculus*, Charles University preprint, Prague 1981
- [PST2] ———, *Simple construction of  $SU(3)$  representations using the  $SU(2)$  projection technique*, Czech. J. Phys. B 35 (1985), 593–601
- [PST3] ———, *Clebsch-Gordan coefficients of  $SU(3)$  with simple symmetry properties*, J. Phys. A: Math. Gen. 19 (1986), 21–28
- [Sh] J. Shapiro, *Matrix representation of the angular momenta projection operator*, J. Math. Phys. 6 (1965), 1680–1691
- [ST] Yu.F. Smirnov and V.N. Tolstoy, *Extremal projectors for usual, super and quantum algebras and their use for solving Yang-Baxter problem*, in: *Selected Topics in QED and Mathematical Physics* (J. Niederle and J. Ficher, eds.), World Scientific, Teaneck NJ 1990, pp. 347–359
- [STK1] Yu.F. Smirnov, V.N. Tolstoy, and Yu.I. Kharitonov, *Method of projection operators and  $q$ -analog of the quantum angular momentum theory. I. Clebsch-Gordan coefficients and irreducible tensor operators*, Sov. J. Nucl. Phys. 53 (1991), 593–605 [Yadernaya Fiz. 52 (1991), 959–980]
- [STK2] ———, *Projection-operator method and the  $q$ -analog of the quantum theory of angular momentum. Racah coefficients,  $3j$ - and  $6j$ -symbols, and their symmetry properties*, Sov. J. Nucl. Phys. 53 (1991), 1068–1086 [Yadernaya Fiz. 53 (1991), 1746–1771]
- [STK3] ———, *Tree technique and irreducible tensor operators for the  $su(2)$  quantum algebra.  $9j$ -symbols*, Sov. J. Nucl. Phys. 55 (1992), 1599–1604 [Yadernaya Fiz. (1992), 2863–2874]
- [STK4] ———, *Tree technique and irreducible tensor operators for the  $su(2)$  quantum algebra. Algebra of tensor operators*, Sov. Phys. Atom. Nucl. 56 (1993), 690–700; [Yadernaya Fiz. 56 (1993), 223–244]
- [SST] D.T. Sviridov, Yu.F. Smirnov, and V.N. Tolstoy, *On the structure of the irreducible representation basis for the exceptional group  $G_2$* , Rep. Math. Phys. 7 (1975), 349–361
- [TV] V. Tarasov and A. Varchenko, *Duality for Knizhnik-Zamolodchikov and dynamical equations*, Acta Appl. Math. 73 (2002), 141–154
- [T] V.N. Tolstoy, *Method of projection operators for unitary groups*, Master’s thesis, Institute of Nuclear Physics, Moscow State University, Moscow 1969
- [T1] ———, *Extremal projectors for reductive classical Lie superalgebras with non-degenerate generalized Killing form*, Uspekhi Mat. Nauk 40 (1985), 225–226 (in Russian)
- [T2] ———, *Extremal projectors and reduced superalgebras over finite-dimensional contragredient Lie superalgebras*, in: *Group Theoretical Methods in Physics*, Vol. I (Yurmala 1985), VNU Sci. Press, Utrecht 1986, pp. 323–335

- [T3] ———, *Fundamental system of the extremal projectors for  $sl(2)$* , in *Group Theoretical Methods in Fundamental and Applied Physics* (Vladivostok 1986), Nauka, Moscow 1988, pp. 258–259 (in Russian)
- [T4] ———, *Extremal projectors for contragredient Lie algebras and superalgebras of finite growth*, *Russ. Math. Surveys* 44 (1989), 257–258
- [T5] ———, *Extremal projectors for quantized Kac-Moody superalgebras and some of their applications*, *Springer Lecture Notes in Physics* 370 (1990), 118–125
- [T6] ———, *Combinatorial structure of root systems for Lie (super)algebras*, in: *Proceedings of the International School Symmetries and Integrable Systems* (Dubna, June 1999), JINR Publications, Dubna 2001, pp. 139–168
- [T7] ———, *Projection operator method for quantum groups*, in: *Special Functions 2000: Current Perspective and Future Directions* J. Bustoz, M.E.H. Ismail, and S.K. Suslov, eds.), Kluwer Academic Publishers, Dordrecht 2001, pp. 457–488 [math.QA/0104045]
- [TIS] V.N. Tolstoy, I.F. Istomina, and Yu.F. Smirnov, *The Gelfand-Tseitlin basis for the Lie superalgebras  $\mathfrak{gl}(n/m)$* , in: *Group Theoretical Methods in Physics*, Vol. I (Yurmala 1985), VNU Sci. Press, Utrecht 1986, pp. 337–348
- [TD] V.N. Tolstoy and J.P. Draayer, *New approach in theory of Clebsch-Gordan coefficients for  $u(n)$  and  $U_q(u(n))$* , *Czech. J. Phys.* 50 (2000), 1359–1370
- [Zh1] D.P. Zhelobenko, *S-algebras and Verma modules over reductive Lie algebras*, *Dokl. Akad. Nauk. SSSR* 273 (1983), 785–788
- [Zh2] ———, *Z-algebras over reductive Lie algebras*, *Dokl. Akad. Nauk. SSSR* 273 (1983), 1301–1304
- [Zh3] ———, *Extremal projectors and generalized Mickelsson algebras over reductive Lie algebras*, *Math. USSR Izv.* 33 (1989), 85–100 [*Izvestiya Akad. Nauk. SSSR Ser. Mat.* 52 (1988), 758–773]
- [Zh4] ———, *An introduction to the theory of S-algebras over reductive Lie algebras*, in: *Representation of Lie Groups and Related Topics*, Gordon and Breach, New York 1990, pp. 155–221

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